THE
MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., AND PROF. E. T. WHITTAKER, M.A., F.R.S.

LONDON:
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY,
AND BOMBAY.

VOL. VII. DECEMBER, 1913. No. 108.

# THE SIMPLE PENDULUM.

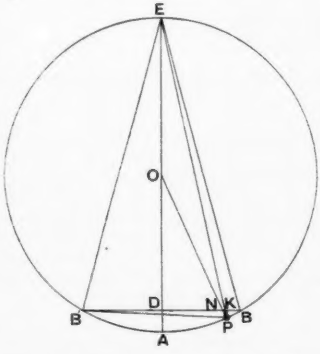
MR. D. K. PICKEN has done well to direct attention* to the unsatisfactory nature of the elementary discussion of the Simple Pendulum, current in the text-books.

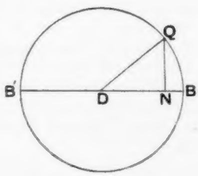
What is usually passed off as a proof is a downright wicked deceit, and deserves the serious attention of the advocate of rigour, when he can spare time to look elsewhere, beyond a fundamental axiom or a convergency test.

The beat of a pendulum is always greater than is given by the formula

$$\pi\sqrt{\frac{l}{g}};$$

but in the ordinary visible oscillation of a clock pendulum a superior limit can be assigned with ease to the beat, giving very close limits between which the real beat must lie; and these limits can be found by elementary geometry.




Take a pendulum suspended at $O$, or the equivalent thread pendulum $OP$, with its centre of oscillation $P$ swinging from $B$ to $B'$ on the arc $BAB'$ of a

* v. *Mathematical Gazette*, vol. vii. p. 173.

circle on the vertical diameter $AE$, with no restriction on the extent of oscillation.

The only dynamical theorem required asserts that the velocity of $P$ is that due to falling vertically the distance $PK$ from $BB'$; and by Euclid VI. C,

If $EP$ cuts $BB'$ in $N$,

(1) $$PK=\frac{PB \cdot PB'}{AE}.$$

(2) $$\frac{PB \cdot PB'}{NB \cdot NB'}=\frac{EA \cdot PK}{EN \cdot NP}=\frac{EA}{EN} \cdot \frac{ED}{EN}=\frac{EB^2}{EN^2}.$$

(3) $$(\text{Vel. of } P)^2=2g \cdot PK=\frac{2g}{AE} PB \cdot PB'$$

$$=\frac{2g}{AE} \cdot \frac{EB^2}{EN^2} NB \cdot NB'.$$

Here $\frac{2g}{AE}$ is the $\frac{g}{l}$, usually denoted by $n^2$;

(4) $$\text{Vel. of } P=n \frac{EB}{EN} \sqrt{(NB \cdot NB')}.$$

Since $EP$ cuts $BB'$ and the circle at the same angle,

(5) $$\frac{\text{Vel. of } N}{\text{Vel. of } P}=\frac{EN}{EP}=\frac{EN^2}{EB^2}.$$

(6) $$\text{Vel. of } N=n \frac{EN}{EB} \sqrt{(NB \cdot NB')}=n \frac{EN}{EB} NQ,$$

if $NQ$ is the ordinate of the circle $BQB'$ on the diameter $BB'$.

By the triangle of velocity,

(7) $$\text{Vel. of } Q=n \frac{EN}{EB} \cdot DQ,$$

fluctuating between $n \cdot DB$ at the end of a beat, and $n \frac{ED}{EB} DB$ at the middle of the swing; so that

(8) $$\frac{\pi}{n} < \text{beat} < \frac{\pi}{n} \frac{EB}{ED}=\frac{\pi}{n} \sec \tfrac{1}{2}\alpha,$$

if $2\alpha$ is the angle of oscillation; and these limits are close together when this angle is small.

In the usual elementary treatment it is assumed that the lower limit may be taken for the beat; the angle of oscillation $BOB'$ (not the arc or chord $BB'$ necessarily) is supposed so small that $P$ and $N$ are undistinguishable; and the velocity of $P$ or $N$ being $n\sqrt{(PB \cdot PB')}$ or $n\sqrt{(NB \cdot NB')}$ is the quality of a simple vibration, making the velocity constant and $n \cdot DQ$ of the satellite point $Q$ on the circle $BQB'$; and $Q$ may be considered the bob of a conical pendulum moving round a horizontal circle, at the level of $A$.

Then $Q$ describes the semi-circle $\pi \cdot BD$ with velocity $n \cdot BD$, while the pendulum bob $P$ moves from $B$ to $B'$, and whatever the extent of $BB'$, the beat is

$$\frac{\pi \cdot BD}{n \cdot BD}=\frac{\pi}{n} \text{ seconds.}$$

We call this a *simple vibration*, and omit the word *harmonic* as redundant and misleading, reserving harmonic to designate an overtone given by a term of the Fourier series, of which the fundamental note is the simple vibration.

The notion of acceleration is not required in this treatment of the pendulum; but the property of the hodograph is employed in Maxwell's *Matter and Motion*, Chapter VII., to show that the uniform motion of $Q$ round its circle requires an acceleration $n^2 . QD$ to the centre $D$, and then the acceleration of $N$ is $n^2 . ND$, as in the vibration of a spring.

The point $P$ would make a simple vibration over the arc $BAB'$, whatever its extent, if the velocity of $P$ was $n$ times the geometric mean of the arc $PB$ and $PB'$, instead of the chord. This may be taken as the case in the oscillation of the balance wheel of a watch, where it is assumed that the resilient couple of the hair spring is proportional to the angle of displacement from equilirbium.

The usual formula gives the lower limit for an invisible beat; but the *circular correction* for the visible beat of a clock pendulum is fundamental in Horological Theory, as in its application, for instance, to the Hope-Jones synchrome system.

The geometric mean of the limits in (8) may be taken as a close approximation for the visible beat of a clock pendulum; giving a beat

$$(9)\qquad \frac{\pi}{n}\surd(\sec\tfrac{1}{2}\alpha)=\pi\sqrt{\frac{l}{g\cos\frac{1}{2}\alpha}}=(1+f)\pi\sqrt{\frac{l}{g}},$$

as if the plane of oscillation was tilted through $\frac{1}{2}\alpha$ from the vertical, in the invisible beat; and then $f$ is called the *circular correction.*

Then to our approximation, where $\alpha$ is small and $f^2$ may be ignored,

$$(10)\qquad 1+2f=\sec\tfrac{1}{2}\alpha=1+\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{4}\alpha.$$

$$(11)\qquad f=\tfrac{1}{2}\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{4}\alpha=(\tfrac{1}{2}\tan\tfrac{1}{2}\alpha)^2=(\tfrac{1}{4}\sin\alpha)^2=\left(\frac{BB'}{4AE}\right)^2=\left(\frac{BB'}{8l}\right)^2.$$

If the invisible oscillation gains one beat in $N$ of the visible motion,

$$(12)\qquad 1+f=\frac{N+1}{N},\quad f=\frac{1}{N},\quad N=\left(\frac{8l}{BB'}\right)^2.$$

Thus for so large an angle of oscillation as 60°, $BB'=l$, and the approximation in (12) makes $N=64$, one second beat is one minute four seconds.

For a gain of one beat in an hour, $N=3600$, $BB'=\dfrac{8l}{60}$, and with $l=39$, $BB'=5.2$ inches.

In the treatment above, the oscillating motion of $P$ on the arc $BAB'$ has been replaced by a continuous motion of the satellite $Q$ round a circle, with a velocity fluctuating only to a slight extent.

Returning to the exact treatment, and denoting the angle $BDQ$ by $\phi$,

$$(13)\qquad EB^2-EN^2=DB^2-DN^2=NQ^2=DB^2\sin^2\phi,$$

$$(14)\qquad \frac{EN}{EB}=\surd(1-\kappa^2\sin^2\phi),\quad \kappa=\frac{DB}{EB}=\sin\tfrac{1}{2}\alpha,\quad \frac{\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\alpha}=\frac{DN}{DB}=\cos\phi,$$

and from (7),

$$(15)\qquad \frac{d\phi}{dt}=n\surd(1-\kappa^2\sin^2\phi),\quad nt=\int_0^\phi\frac{d\phi}{\surd(1-\kappa^2\sin^2\phi)},$$

and this is Legendre's *first elliptic integral*, denoted by $F\phi$ or $F(\phi,\kappa)$, and tabulated in his *Fonctions elliptiques*, Vol. II. Table IX.; and here the student of Dynamics makes his first acquaintance with the Elliptic Function (*Horological Journal*, September, 1913).

In visible oscillation of finite extent, the beat

(16) $$T=\sqrt{\frac{l}{g}}\int_0^{\pi}\frac{d\phi}{\sqrt{(1-\kappa^2\sin^2\phi)}}=2K\sqrt{\frac{l}{g}}=\frac{K}{\frac{1}{2}\pi}\pi\sqrt{\frac{l}{g}},$$

(17) $$1+f=\frac{K}{\frac{1}{2}\pi},\qquad K=\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-\kappa^2\sin^2\phi)}};$$

and $K$ is tabulated by Legendre in Table I. to every tenth of a degree in the modular angle $\frac{1}{2}\alpha$.

Expanding in powers of $\kappa^2$, and integrating,

(18) $$\frac{K}{\frac{1}{2}\pi}=1+\tfrac{1}{4}\kappa^2+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2\kappa^4+\ldots,\qquad f=\tfrac{1}{4}\kappa^2\ldots,\qquad N=\frac{4}{\kappa^2},$$

as above in (11), (12).

For such a large swing as through 60°, from V to VII on the clock face,

$$\tfrac{1}{2}\alpha=15^\circ,\qquad N=\frac{4}{\sin^2 15}=\frac{8}{1-\cos 30^\circ}=\frac{16}{2-\sqrt{3}}=16(2+\sqrt{3}),$$

say $N=60$, instead of 64 as before, a gain of one second beat in a minute.

G. Greenhill.

## NOTE ON FINDING PRIME NUMBERS.

It is sometimes very difficult to recognise prime numbers at sight, or to discover them without laborious operations, and mathematicians from the remotest times have endeavoured with varying success to discover suitable methods for the purpose.

In the present note I will assume that the junior student is familiar with the Sieve of Eratosthenes, and that he is required to make a complete list of primes and composite numbers between two limits such as 1601 to 1639.

The following is an elementary method that is quite within his powers. Take slips of squared paper of any breadth, $p$ units long, and divided into $p$ squares, where $p$ is the factor or modulus to be considered. The slips are periodic. The first is always black and the $p-1$ other squares white. It is sufficient to know the position of the black square for each modulus in the space considered, but if the black squares are all outside the limits on a slip, it is easily seen that this slip is not required. I have constructed a table for the first million, giving these results very easily, even for very large numbers. But for the small numbers with which they have to deal the students will not want this. For instance, they will divide the smallest of the numbers by the modulus, and they will see at once what remainder must be added to the number for it to be divisible by this modulus.

*Example.* 1601 divided by 17 gives remainder 3; ∴ 14 must be added to 1601 to have the smallest multiple of 17 within the limits imposed; *i.e.* 1615 will be the first black square of the periodic slip of modulus 17. It is easily seen that 29, 31, 37, and 41 give no black square within the given limits, so that these slips are not required, and we have the following diagram, in which a column with all its squares gives all the primes correctly.

Thus we can read off the primes 1601, 1607, 1609, 1613, 1619, 1621, 1627, 1637. The remaining 12 numbers are composite, and one at least of the factors of each is the modulus in brackets on the left hand of the diagram.

Thus $$1617 = 3 \times 7^2 \times 11\ ;\ 1635 = 3 \times 5 \times 109, \text{ etc.},$$
numbers completely factorised by the method as long as the necessary and sufficient condition is taken, *i.e.* the appropriate number of periodic slips. For the benefit of our younger readers, it may be noted that for a number $N$, we must take all the moduli $<\sqrt{N}$, but for numbers of special forms there are many important simplifications possible.

| *Numbers* | 1601 | 1603 | 1605 | 1607 | 1609 | 1611 | 1613 | 1615 | 1617 | 1619 | 1621 | 1623 | 1625 | 1627 | 1629 | 1631 | 1633 | 1635 | 1637 | 1639 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (mod. 3) | | | ■ | | | ■ | | | ■ | | | ■ | | | ■ | | | ■ | | |
| ( „ 5) | | | ■ | | | | | ■ | | | | | ■ | | | | | ■ | | |
| ( „ 7) | | ■ | | | | | | | ■ | | | | | | | ■ | | | | |
| ( „ 11) | | | | | | | | | ■ | | | | | | | | | | | ■ |
| ( „ 13) | | | | | | | | | | | | | ■ | | | | | | | |
| ( „ 17) | | | | | | | | ■ | | | | | | | | | | | | |
| ( „ 19) | | | | | | | | ■ | | | | | | | | | | | | |
| ( „ 23) | | | | | | | | | | | | | | | | | ■ | | | |

The slips may be used for any series of numbers; all that changes is the number of slips required, which increases with the size of the numbers under consideration. With, at most, 24 slips we have both all the primes $<10200$, and the factors of all the intermediary numbers. The square need not be blackened—it is sufficient to give it a distinctive mark by asterisk or letter. The results are clearer still if we use slips of coloured paper, or better still, wooden or metal rules such as are required in more elaborate investigations.

The advantage of the present method consists in its appealing to the eye and in the speed with which the results are obtained. I hope this note will call the attention of our younger readers to the fascinations of the study of integers, and if the occasion arises it would give me great pleasure to organise on their behalf a competition on the subject. At any rate there is no harm done in asking teachers to follow the example of many of their colleagues and to draw the attention of their pupils to this new and simple method of attacking the problem of finding prime numbers.

A. Gérardin.

32 quai Claud le Lorrain,
Nancy.

## THE TEACHING OF NUMERICAL TRIGONOMETRY.*

Now that there is a tendency to cut out a good deal of useless matter in arithmetic and algebra, and to spend less time over formal geometry, it is possible, and I think profitable, to introduce boys quite early to a course of numerical trigonometry. For the last fifteen years, I have seen some such course as is here described taught successfully to boys of thirteen and fourteen years of age. The inclusion of certain types of question in arithmetic and algebra courses—such, for instance, as those wonderful creations that you will find in chapters on vulgar fractions—is, I suppose, justified on the ground that they call for patience, orderly arrangement of work, and accuracy. But scope can

* Part of a paper read to the London Branch of the Mathematical Association.

be found for the exercise of all these qualities in numerical problems which arise in connection with other branches of their work—geometry, mechanics, physics, etc.—and the range of such problems is greatly increased if the boy has an elementary knowledge of trigonometry. Moreover, the work involved in solving triangles is often fairly complicated and must be done carefully and arranged in an orderly manner, otherwise it is liable to end in disaster. There is a certain amount of drudgery involved, but it is drudgery with some interest attached to it.

Owing, I think, to the fact that the teaching of the subject used to begin from the wrong end, trigonometry was in my youth, and still is, I suppose, in many schools, looked upon as a luxury for the select few. As a matter of fact, up to a certain point, the ordinary boy with no particular mathematical bent finds it easier than any other part of mathematics. Many boys who will never make much headway with demonstrative geometry, for instance, become quite respectable performers in trigonometry. I mean what Continental writers call trigonometry as distinguished from goniometry. I remember a familiar phrase which used to occur in examination syllabuses: "Trigonometry up to Solution of Triangles." The trigonometry I am now referring to might almost be said to begin and end with solution of triangles, and the tools are only provided as they are required. I suppose everyone here learnt his trigonometry under the old scheme, and perhaps some of you remember the shock you received when six trigonometrical ratios with appalling names were suddenly fired off at you. Before you had recovered from the shock, you plunged headlong into a set of exercises called identities, which to the mathematically inclined were amusing puzzles, but to the ordinary person were unfathomable mysteries calculated to choke him off at the very outset. Then again, one was not allowed to begin the subject until one had done quite a lot of Algebra and six books of Euclid, the reasons being, I suppose, that the constancy of the ratio $\frac{a}{b}$ in triangles with the same angles must be formally established before the trigonometrical ratios could be defined. For the modern boy who is born drawing to scale, the fact that equiangular triangles have the same shape requires no proof. Why worry about one which, even when he has got it, does not convince him? The ratios should be introduced gradually, and their meaning driven home by easy problems which seem to be much more helpful for this purpose than identities. The more varied the problems the better, but the variety must of course depend on the amount of knowledge the boy has of other branches, such as geometry and mechanics. It think it is better, until the general triangle formulae are reached, to keep to acute angles. I generally begin with the tangent as being connected with the easiest kind of height and distance problem, and it should be in connection with some such problem, solved by scale drawing, that the idea of tangent of an angle is introduced. Suppose a class starting trigonometry to be confronted with the following problem, which they may have met in carrying out some elementary field work: "Find the height of a tower given that its elevation at a distance of 120 feet is 35°." They will each draw a right-angled triangle $OMP$. The actual lengths of $OM$ representing 120 feet and $MP$ representing the height of the tower will be different with different boys, but all will get more or less the same result for the height of the tower. It should not be difficult to elicit a suggestion that this is so because $\frac{MP}{OM}$ is the same in all their figures, and indeed in all right-angled triangles in which $\angle MOP=35°$. Therefore, if somewhere they had a record of what this

ratio was they could solve the problem without drawing a figure, and probably gain both in time and accuracy. To emphasise this idea, let every boy draw an angle of 35°, $AOB$, take any point $P$ in one arm and draw $PM$ perpendicular to the other. Then let him measure carefully $OM$, $MP$ and work out $\frac{MP}{OM}$ correct to two decimal places. They will all have different lengths for $OM$ and $MP$, but should obtain the same value for $\frac{MP}{OM}$. This number (·70) is called the tangent of 35°, and a definition is suggested for the tangent of any acute angle:—From any point $P$ in one arm of the angle $AOB$ draw $PM$ perpendicular to the other arm; then $\frac{MP}{OM}$ is called the tangent of the angle: or we may say that the tangent of the angle is the multiplier which converts $OM$ into $MP$.

The tangents of various angles will now be found, and the class will see how the work is simplified by taking $OM$ of a convenient length. They should also see the advantage of large figures in keeping down errors. Both these suggestions should be extracted from the boys themselves. It will be obvious (by keeping $OM$ the same length in all figures, for instance) that every acute angle has a different tangent. Boys should now make the acquaintance of a table of natural tangents, and it may occur to some one to ask why he could not get four decimal places from his drawing. Interesting arithmetical exercises may be set round this point; *e.g.* if $OM$, $MP$ are found by measurement to be 5·00″ and 3·50″ respectively, and if there is a possible error of ·01″ either way in each measurement, how many figures can we be sure of in the calculated value of tan 35°?

When a boy can read off the tangent of any angle from the tables, the definition might be presented in a slightly different form. $ABC$ is a right-angled triangle in which $C$ is the right angle. Then, of the two sides which contain the right angle $a$ is opposite to $A$ and $b$ is adjacent to $A$; $b$ is opposite to $B$ and $a$ is adjacent to $B$; and it may be said of either of the angles $A$ and $B$ that its tangent is $\frac{\text{opposite side}}{\text{adjacent side}}$. It is important that a boy shall be able to say not only that $\frac{a}{b}=\tan A$, but also $a=b\tan A$, and he should have practice in reading off such relations from figures in which use is made of right-angled triangles whose sides are not parallel to the edges of the paper. For instance, let $PQR$ be a triangle with a right angle at $Q$, and let $QS$ be perpendicular to $PR$—then $PQ=QR\tan R$, $PS=SQ\tan R$, $QS=SR\tan R$.

He is now ready for his first essay in solving right-angled triangles. The arithmetic should not be too heavy at first, and this remark applies to all cases where a boy is dealing with a new idea. Even at this stage there will generally be more than one way of attacking a problem, and a boy should be encouraged to choose the least troublesome one, though, if he fails to do this, he should be made to go through with his own plan. A simple example will show what I mean. In the triangle $ABC$, $C=90°$, $A=50°$, $b=10$; find $a$. We can say either $a=b\tan 50°=10\times 1{\cdot}1918$ or $b=a\tan 40°$;

$$\therefore\ a=\frac{b}{\tan 40°}=\frac{10}{{\cdot}8931},$$

and the first is obviously easier.

Some easy problems on right-angled triangles, isosceles triangles, heights of towers, etc., can now be done. A few examples are given.

1. An angle of a rhombus is 40° and the shorter diagonal is 8 cms. Find the other diagonal.

2. From a ship $A$ bears N. 55° W. When she has sailed 10 miles due W., $A$ is due N. How far is she from $A$ at the second observation?

3. From points on opposite sides of a tree 50 ft. high, two men observe its angles of elevation to be 25° and 35°. How far are they apart?

4. Find the perimeter of a regular pentagon circumscribed to a circle of 5 inches radius.

5. From the top of a cliff 120 feet above the water, the angles of depression of two boats lying in the same direction are 8° and 30°. Find the distance between the boats.

The next thing is to see that since every acute angle has a different tangent, therefore, if the tangent is given, the angle is determined. A few problems such as "construct and measure with a protractor the acute angle whose tangent is ·36" should be done, and a little practice given in reading off from the table angles with given tangents. Problems will follow, such as:

1. Find the altitude of the sun if the length of the shadow of a man 6 ft. high is 8 ft. 3 ins. long.

2. Two forces 2 and 3 lb. wt. act at right angles. Find the magnitude and direction of the resultant.

3. A crane has a vertical post 8 feet long, a tie 8 feet long and a jib 12 feet long. Find the angles made by the post with the tie and jib.

4. Find the vertical angle of a cone if the diameter of the base is 10 cms. and the vertical height 15 cms.

The sine and cosine are now introduced in an exactly similar way. It might be interesting to create a demand for them by setting an inconvenient problem to be done by the use of the tangent, *e.g.* "In a triangle $C=90°$, $c=5$, $A=40°$; find $a$." The only available method would be something like this:

$$b=a\tan 50°=a\times 1{\cdot}1918$$

and

$$a^2+b^2=c^2;$$

$$\therefore\ a^2(1+1{\cdot}420)=25,$$

$$\therefore\ a=\frac{5}{1{\cdot}556}=3{\cdot}21.$$

[The tables of squares and square roots are useful here, and a reciprocal table will save the last division.]

The advantage of being able to express $a$ directly in terms of $c$ is obvious, and the sine and cosine come in naturally. A course of easy exercises on these ratios should be given on the same lines as those on the tangent.

1. A man walks 100 yards up a slope of 5°. How high is he above his starting-point?

2. A body weighing 10 lbs. is on an inclined plane whose angle of slope is 20°. What are the resolved parts of the weight along and perpendicular to the plane?

3. Find the length of a chord which subtends an angle of 110° at the centre of a circle of radius 8 cms.

4. In a simple engine the connecting rod is $3\frac{1}{2}$ feet long and the crank 1 foot long. Find the angle between the crank and the line of dead centres at half-stroke.

5. A side of a triangle is 4 inches and the opposite angle 70°. Find the radius of the circumcircle.

6. Find the resultant of two equal forces acting at an angle of 40°.

7. Find the area of a parallelogram whose sides are 12 cms. and 8 cms. and included angle 55°.

Any right-angled triangle can now be solved. The arithmetic should still be kept within bounds and the habit of checking encouraged from the start. By actual drawing a boy should convince himself that the parts of a right-angled triangle are determined, if in addition to the right angle he is given either (i) any two sides or (ii) any side and any angle. He will expect, therefore, to be able to calculate the remaining parts in any of these cases.

For example, given $C=90°$, $A=54°$, $a=4$, solve the triangle.

(i) $B=36°$.

(ii) $b$ will be found from $\tan A$ or $\tan B$, *i.e.* either $b=4\tan 36°$ or $b=\frac{4}{\tan 54°}$, the former for choice; this gives $b=2·906$.

(iii) $c=\frac{4}{\sin 54°}=\frac{4}{·8090}=4·944$.

(iv) Check. $c^2=4·944^2=24·44$, $a^2+b^2=4^2+2·906^2=16+8·445=24·445$.

Numerous problems of all kinds can now be done, and these afford plenty of opportunity for a boy to use his wits in making the best use of the weapons at his disposal.

J. W. MERCER.

(*To be continued.*)

## THE CALCULUS AS AN ITEM IN SCHOOL MATHEMATICS.*

FEW subjects deserve attention better than the problem of teaching higher mathematics to the average student.

Any man who does work in the world will some day, even though he may be a classical scholar investigating variants in manuscripts, be the worse for not knowing more mathematics; and if he is not too closely limited by his environment he may even realise this fact. A goodly proportion of the work of the world must be done by quite ordinary folk. Something will be gained if the average student is helped even a little by being enabled to use the calculus and to think of its machinery as an engineer thinks of a drill, and not as a West Coast native thinks of a Ju-ju.

An illustration may be taken from the Astronomer Royal's recent text-book. A star of the third magnitude is very much brighter than a star of the fourth magnitude. But the fourth-magnitude stars are more numerous than the third-magnitude stars, so that in fact we receive more light from the fourth-magnitude than from the third-magnitude stars. To raise all fourth-magnitude stars to the third magnitude would increase our light more than to raise all third-magnitude stars to the second magnitude.

* A paper read in February, 1911, at the London Branch of the Mathematical Association. Acknowledgments are due to Messrs. G. St. L. Carson and C. Godfrey, Professor A. Lodge and Mr. W. L. Sheppard, for suggestions and criticisms verbal and by correspondence. But these gentlemen must not be held responsible for the opinions expressed. Further experience has led the author to attach still greater importance to the arithmetical preparation and, on the other hand, to believe that if the Calculus is to be carried beyond this preliminary stage, formal differentiation must be practised until facility is attained.

## THE CALCULUS AT SCHOOL 25 YEARS AGO.

What is the position from which we start?

Twenty-five years ago the treatment of the calculus in schools was well settled. Only a few of the best schoolboy mathematicians studied the subject, usually in Todhunter's books.

No one who has reached middle age will ever speak other than gratefully of Todhunter's books—so scrupulously accurate in numerical detail. Nor should it be forgotten that Dr. Todhunter had specially at heart the needs of the solitary student, and that to the close of his life, as I have been told, he was always ready to answer letters from students who had encountered difficulties. Boys reared on Todhunter could differentiate and they could integrate, and if, subsequently (there is much virtue in an if), they learned to apply their knowledge to problems of mechanics and physics, this facility in the formal treatment did them yeoman service. At the present time there is a risk that in some quarters manipulative skill may be undervalued and neglected.

Any one, perhaps, not even excepting a professional mathematician, who tries to apply his mathematical knowledge to a real and novel problem, is sure not to take the shortest or neatest path, and generally finds himself involved in unwieldy calculations. In these circumstances some power of patient handling of rather clumsy formulae will not be the least valuable relic of his schoolboy mathematics.

The attack on the apparently well-established routine of beginning the calculus in this manner with boys of 17 or 18—and then only in exceptional cases—came from two distinct quarters. One attack was delivered from the mathematical side—from the rigorists who complained of unguarded generalities.

A typical criticism may be given in dialogue form.

You have shown how to differentiate a function of $x$?

Certainly.

Then will you please find the differential coefficient with respect to $x$ of $y$, defined as the greatest integer in $x$?

A celebrated review in the *Bulletin* (vol. i. p. 217) by Miss C. A. Scott may be cited as a specimen of destructive criticism of the older school of text-books.

The other attack came from the teachers of physics, engineering, and applied mathematics. They pointed out that considerable manipulative skill might coexist with ignorance of the meaning and principles of the subject. The stickler for manipulative skill did not, perhaps does not, always realise that a candidate can be taught to say "$-2 \sin 2x$" when an examiner says "$\frac{d \cos 2x}{dx}$?" with about as much meaning or propositional assertion as in a certain game in which "snip" is the correct reply to "snap."

The teachers of physics and engineering further complained that the only applications dealt with were to higher plane curves, and that the investigation of the properties of these curves was apt to degenerate into obscure scandal about the circle of curvature at some point also going through the origin. They demanded, firstly, that the calculus should be more widely taught and at an earlier age, and secondly, that from the outset the more immediate applications to elementary mechanics and physics should be dealt with.

Unfortunately—under the pressure of these two distinct attacks—the elementary treatment of the calculus has tended to bifurcate into distinct types, and we are threatened in the calculus with a repetition of that objectionable antagonism between the academic and the practical from which, in mechanics, there are some signs of coming liberation.

Widely different introductions to the calculus have been framed by the rigorist and by the engineer. The rigorist puts minute logical precision first—the practical man asks us to dwell on the utility of the calculus. Of the two the rigorist, in relation to the schoolboy, is by far the more astray. For more than a hundred years after the formal development of the calculus the most eminent mathematicians in Europe ignored these difficulties with which the rigorist wants to perplex beginners. We, as teachers, are just learning that you cannot teach people to generalise by throwing ready-made generalisations at them—that a grip on the concrete facts must precede a critical analysis of them—that there is no worse mistake than to jump before you come to the fence. A pupil's comment on a well-known text-book was: "If there is a real difficulty he says nothing about it—he passes it over—but about things which are not at all difficult he jaws for pages." This criticism is doubtless highly subjective, but perhaps not therefore less significant.

But it is needless to say more. That third-rate critical work is of very low value in comparison with constructive work, and that the capacity for the appreciation of subtle analysis is usually developed late, will not be disputed by any experienced teacher. It is, however, quite a mistake to suppose that because a man's future profession is to analyse coal-gas residues a distinct flavour of coal gas ought to be communicated to his school atmosphere.

A genuine interest in the philosophic and scientific basis of mathematics is sometimes found in quarters where it would not have been sought for hopefully. But though the practical man has sometimes forgotten the true function of education—forgotten the supreme importance of keeping out of the rut as long as may be—he has been right in asking us to deal first with concrete and familiar problems, and to emphasise by appropriate illustrations that mathematics is the widest and broadest element in a liberal education—the only subject which has points of contact with every human interest.

In trying to adopt the sound portions of both schemes without blindly following either, the teacher requires before all things a sense of proportion and a power of seeing both sides of the question, for although in this subject the fine distinctions of the rigorist are not mere subtleties but have more than once found their justification in guarding against error in a physical investigation, yet it is easy, in the spirit of him who praised the wisdom of Adam in giving to each animal its right name, to put the emphasis on nomenclature rather than on principles. Attention to names may take the place of a power over things—verbal precision degenerate into verbose preciosity—and we do not escape from the blessed word Mesopotamia because it is not always spelt the same way.

But on the other hand rigour means clearness. A conclusion cannot really follow better from an argument that does not prove it than from an argument which does. Students' difficulties are due, more often than is sometimes supposed, to a creditable reason—their logical insight tells them that there is a gap in the argument as presented. Not only so, but there are strong reasons in favour of using at any rate some words with definite meanings. Illustrations may easily be found in technical handbooks and engineering journals of the unhappy fact that the laxity of expression and looseness of reasoning which were commented on in Todhunter and Pearson's *History of the Theory of Elasticity* are not yet abandoned by the "practical man."

But these vague generalities are all very well. Admitting that we must be as academic and also as practical as is consistent with our

main object of being convincing, that we must not let the principle be completely obscured by the details of the argument, the question is: How is the calculus, whose history is that of the mathematical genius of two thousand years and whose applications range over every known science, to be brought into the limits of space and time at the disposal of a schoolboy?

In more definite form the problem is whether the calculus can be made an element in the normal scheme of mathematics of a secondary school, so that a boy who reaches a respectable position in the school by the age of 16 or 17 will in the ordinary course have acquired a useful knowledge of the subject. If this is to be the case something must of course be sacrificed. We cannot at a stroke bring the average boy of 16 to the level of the exceptional boy of 19. What are we to try for, and is what can be done worth doing?

Some of us have had a view—all have read—of an experiment in which a photograph is taken daily of a plant. The resulting series of photographs are shown by a cinematograph, with the result that the spectators see in five minutes the more obvious features of a process of growth which occupied weeks. Just so it is the task of a teacher of the calculus to traverse in a year or so the country which it has taken mankind two thousand years to explore in detail. It is the object of this paper to argue that in broad outline there is one way—one fairly definite sequence—to which the teaching of the subject ought with variations to conform. The teacher's main concern, as Mr. Daniell happily expresses it, is with the expanding outlook of his pupils on the world. No still-life picture of the plant can replace the view of the growth of bud and leaf, blossom and fruit. The teacher must not be led away by the fatal error that it is his mission to present to the beginner the subject as it stands to-day.

In coming to details after this lengthy preliminary the opinions expressed are inevitably personal. Even, however, if nearly all hearers of a paper find something in it to disagree with, it may serve some purpose in arousing discussion.

## PRELIMINARY WORK.

Returning for a moment to the illustration already used, just as the plant has gone through a period of growth before anything visible emerges above ground, so the study of the calculus should be commenced long before the word itself is used.

Very early in arithmetic attention should be called to the utility of differences. The compilation of a table of squares gives as good an illustration as any.

| Number. | Square. | Step. |
|---|---|---|
| 1 | 1 | |
| | | 3 |
| 2 | 4 | |
| | | 5 |
| 3 | 9 | |
| | | 7 |
| 4 | 16 | |
| | | 9 |
| | | 11 |
| | | 13 |
| | | etc. |

| Number. | Square. | Step. |
|---|---|---|
| 100 | 10000 | |
| | | 201 |
| 101 | 10201 | |
| | | 203 |
| | | 205 |
| | | etc., |

and later on

| Number. | Square. | Step. |
|---|---|---|
| $x$ | $x^2$ | |
| | | $2x+1$ |
| $x+1$ | $x^2+2x+1$ | |
| | | $2x+3$ |
| | | etc. |

As a labour-saving device the value of considering the step or increment becomes obvious. When the study of the graphs of algebraic expressions is commenced the same method of working with the increment will save much of that tedious and brutalising arithmetic which Dr. Filon, in a communication which you will all recall, denounced as one of the vilest accompaniments of graphical work.

## GRAPHS.

To digress for a moment. Many people have been disappointed to find that the enormous amount of time apparently spent on graphs has not yielded as a harvest a general appreciation of the notions of functionality, continuity, and so forth.

But if we look at the facts from the boy's point of view, which for this purpose is the right point of view, can we be surprised? He has spent fifteen minutes in tedious and dull arithmetic—three minutes in the tedious plotting of points. Then he is ready to start on another example without wasting time. I see here, fifteen minutes on arithmetic, three minutes plotting. What time *has* the boy spent on studying graphs? None. If we can save time on the arithmetic and devote it to a careful drawing of the curve—a figuring of the scales, a free use of colour—so as to produce a work of art, five minutes' opportunity to look at the finished work, we may be entitled to say that the boy has been studying graphs. Time spent on devising a suitable brief and expressive heading is exceedingly well spent.

The English lad who devised the advertisement of the London tube railways:

Underground Quickest way
Anywhere Cheapest fare

and the American lawyer who received $1200 for drafting the notice: "Stop, Look, Listen" for the level crossings of the railways in the States earned their money well. The English firm above one of whose gates appears in imperishable metal the legend: "No way in at this entrance," and the railway companies who do not really wish to insist that "Passengers must cross the line by the bridge" were less happily inspired. In art, I believe that the painter who printed: "This is a lion" underneath his picture, did not succeed in founding a school of painting. Let his name, if it be known, be commemorated as a founder of the true school of graphs. Indeed, a student is often engaged in what is to him tedious and absorbing arithmetic or algebra when his teacher thinks that mechanics or economics is being studied. A teacher ought to be able to give his pupils a little leisure to contemplate results quietly, even though in some cases this may lead to a patent waste of five minutes, instead of a latent waste, to which no one would object, of the whole hour.

C. S. Jackson.

(*To be continued.*)

# THE TEACHING OF EASY CALCULUS TO BOYS.

The majority of the boys who enter the Technical Day School of the Borough Polytechnic Institute come from elementary schools, a few only come from secondary schools. The general age of entry is about 13 years, and the boys are about up to the level of the VIIth standard of a good Council School.

As far as we are able, we select for admission boys who have a mechanical or constructive bias. All the boys who enter are expected to take the full three-years' course of work. At the end of that time they go into engineering workshops.

They study in succession the regular geometrical solids, with the lines, surfaces and angles which we meet with in connection with them. Mathematical operations and methods of working are acquired and made use of as the need for them arises, and the geometrical properties are observed and noted as they occur. Thus we usually commence with the geometry and mensuration of the cube, which involves the straight line, right angle and square, and then take in succession the various prisms and pyramids, the cylinder and the cone. Upon commencing the third year's work of the course the boys have an acquaintance with the equation of the straight line, the manipulation and solution of a simple equation, and the use of Mathematical Tables; and it is at this point that the first introduction to the methods of the Calculus takes place.

The third year's work of the course deals with the parabola, cubic curve, sine curve, and the logarithmic curve, each being taken both graphically and numerically. The calculus work, which is of the simplest kind, is taken as it is naturally required in connection with the above curves, and has a very strong bias on the graphical side. It consists of differentiation and integration of the equations of these curves, finding areas of figures contained by the curves, volumes of solids of revolution, maximum and minimum values and sometimes finding centre of gravity.

The first introduction to the methods of the calculus is the showing that $\frac{\delta y}{\delta x}$ measures the slope of a line, and that in a straight line $\frac{\delta y}{\delta x}$ is constant, while in the parabola the value of $\frac{\delta y}{\delta x}$ is not constant.

This is shown in three ways:

(1) Numerically (by finite differences).

(2) Graphically.

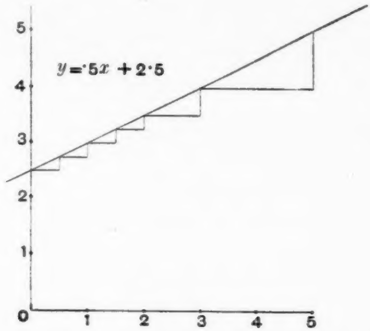
(3) In general terms (*i.e.* algebraically).

(1) The equation of a particular straight line is taken, and by setting out in tabular form coordinate values of $x$, $y$, $\delta x$, $\delta y$, and $\frac{\delta y}{\delta x}$, it is shown that the value of $\frac{\delta y}{\delta x}$ is constant.

$$y = \cdot 5x + 2\cdot 5.$$

| $\delta x$ | $x$ | $y$ | $\delta y$ | $\frac{\delta y}{\delta x}$ |
|---|---|---|---|---|
| | 0 | 2·5 | | |
| ·5 | | | ·25 | ·5 |
| | ·5 | 2·75 | | |
| ·5 | | | ·25 | ·5 |
| | 1 | 3·0 | | |
| ·5 | | | ·25 | ·5 |
| | 1·5 | 3·25 | | |
| ·5 | | | ·25 | ·5 |
| | 2·0 | 3·5 | | |
| 1· | | | ·5 | ·5 |
| | 3 | 4·0 | | |
| 2· | | | 1·0 | ·5 |
| | 5 | 5·0 | | |

This is then represented by a graph in which the lines correspond to the numbers in the preceding tabular form.



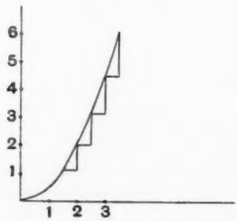

Then each boy works such an exercise independently, choosing his own equation. They thus verify, both graphically and numerically, that in a straight line the value of $\frac{\delta y}{\delta x}$ is constant.

As a result of this we have a number of independent tests, all of which show that the $\frac{\delta y}{\delta x}$ of an equation of the form $y=a+bx$ is constant.

*Note.*—They may have found out the general rule.

The next step is to find whether this is true generally. This is shown by selecting a particular point $(x, y)$ on the line $y=a+bx$, and taking points $(x-\delta x, y-\delta y)$ and $(x+\delta x, y+\delta y)$ on each side of it;

then $$y-\delta y=a+b\,(x-\delta x)$$
and $$y+\delta y=a+b\,(x+\delta x);$$
$$\therefore\ \frac{\delta y}{\delta x}=b.$$




A curve of the form $y=k\,.\,x^2$ is next taken (*e.g.* $y=\cdot5x^2$), and, as before, coordinate values of $x$, $y$, $\delta x$, $\delta y$, and $\frac{\delta y}{\delta x}$ are set out in tabular form, and it is shown that in this case the value of $\frac{\delta y}{\delta x}$ is not constant.

$y = \cdot 5x^2$.

| $\delta x$ | $\delta x_1$ | $x$ | $y$ | $\delta y_1$, $\delta y_2$ | $\frac{\delta y}{\delta x}$ | |
|---|---|---|---|---|---|---|
| | | 1·5 | 1·125 | | | NOTE : |
| 1 | ·5 | 2 | 2 | ·875 | 2 | $x=2$ ; $x-\delta x_1=1\cdot 5$, $x+\delta x_1=2$. |
| | ·5 | 2·5 | 3·125 | 1·125 | | $y-\delta y_1=1\cdot 125$, |
| 1 | ·5 | 3 | 4·5 | 1·375 | 3 | $\delta y_1=\cdot 875$, |
| | ·5 | 3·5 | 6·125 | 1·625 | | $\delta y_2=1\cdot 125$. |

$\delta x=2\,.\,\delta x_1$ ; $\delta y=\delta y_1+\delta y_2$.

(2) The curve is plotted, and it is shown from the graph that the successive values of $\frac{\delta y}{\delta x}$ are not equal, but increase as $x$ increases. (In this case they may or may not discover the general rule. Then each boy repeats the above exercise, choosing his own parabola, and verifying the result, as was done in the case of the straight line, and we, in this case also, have now a number of independent results showing that in curves of the form $y=k\,.\,x^2$ the value of $\frac{\delta y}{\delta x}$ is not constant.

(3) The general case is taken. Here, as with the straight line, points on the curve $\{(x-\delta x_1), (y-\delta y_1)\}$ and $\{(x+\delta x_1), (y+\delta y_2)\}$ are taken on either side of the point $(x, y)$, and from the equations

$$(1)\quad y-\delta y_1=k(x-\delta x_1)^2,$$

$$(2)\quad y+\delta y_2=k(x+\delta x_1)^2,$$

we obtain $$\frac{\delta y_1+\delta y_2}{2\,.\,\delta x_1}=2kx \quad\text{or}\quad \frac{\delta y}{\delta x}=2kx\,;$$

in which $$\delta y=\delta y_1+\delta y_2 \quad\text{and}\quad \delta x=2\,.\,\delta x_1\,;$$

$\therefore$ the value of $\frac{\delta y}{\delta x}$ depends upon the value of $x$, and is not constant.

The boys then work out a table of coordinates, setting it out in the same way as that given above. Then they (1) plot $y$ against $x$ and obtain a parabola; (2) Plot $\frac{\delta y}{\delta x}$ against $x$ and obtain a straight line; (3) Given the curve (1) they obtain the derived straight line by graphic differentiation, without any reference to the table of coordinates.

A good many hints and some help will be needed here at first; dividers should be used to take off the increments of $y$ (take $\delta x=1$ in the first case).

The straight line thus obtained by graphic differentiation should be compared with that obtained by plotting $\frac{\delta y}{\delta x}$ against $x$.

When enough exercises of this type have been worked, and the equations of the derived straight lines written down, then the reverse exercises should be worked, viz., given $\frac{\delta y}{\delta x}=2kx$, find the relation between $y$ and $x$.

(1) Coordinate values of $\frac{\delta y}{\delta x}$ and $x$ are given in tabular form, and the corresponding values of $\delta y$, $y$ and $\delta x$ are obtained.

$$\frac{\delta y}{\delta x}=2x.$$

| $\delta x$ | $x$ | $y$ | $\delta y$ | $\frac{\delta y}{\delta x}$ |
|---|---|---|---|---|
| 1 | 2<br>2·5<br>3 | 4<br>... | ... | 5 |
| 1 | 3·5 | ... | ... | 7 |
| 1 | 4<br>4·5<br>5 | ... | ... | 9 |

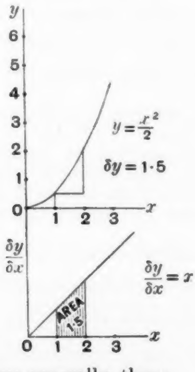
*Note.*—The boys will probably discover that unless one value of $y$ is given they can commence equally well with any value, but I do not go into that question at this point, but just give them a value of $y$ to start with.

(2) Given the straight line $\frac{\delta y}{\delta x}=2kx$, obtain the integral curve by graphic integration. Very little explanation is needed here; the boys soon find out how to reverse the successive steps of the differentiating process.

Dividers are generally used here, and also the method of using a strip of paper and marking the values of $\frac{\delta y}{\delta x}$ in succession along it, and thus automatically adding them together, is used. Plenty of exercises should be worked, so that the method becomes quite familiar, and the various steps of the process should be illustrated numerically and followed out on a table of coordinates.

The next step is to point out the relation between these two curves, viz., that an area contained by the derived curve is given by the length of an ordinate of the integral curve.

This is made clear by actual illustration from a diagram and counting up of squares and numerical calculations.




It is then shown to be true generally, thus:

$$\text{Area}=\frac{\delta y}{\delta x}\times\delta x=\delta y,$$

and $\delta y$ is the difference between the ordinates at $x_1$ and $x_2$.

$\therefore$ $\delta y$ represents the area of a portion contained by the curve, the $x$-axis and the ordinates at $x_1$ and $x_2$.

And this is true for each successive $\delta y$, whence it follows that the whole of a larger area is given by the sum of a series of $\delta y$'s.

It is then shown, both from the diagram and from a tabular form, that the sum of any number of terms in the $\delta y$ column can be obtained (without actual addition) by reference to the $y$ column, *i.e.* that the difference between two terms in the $y$ column gives the sum of the intervening terms in the $\delta y$ column.

| $\delta x$ | $x$ | $y$ | $\frac{\delta y}{\delta x}$ |
|---|---|---|---|
| | 4 | 16 | |
| 1 | 4·5 | ... | 9 |
| | 5 | 25 | |
| 1 | 5·5 | ... | 11 |
| | 6 | 36 | |
| 1 | 6·5 | ... | 13 |
| | 7 | 49 | |
| 1 | 7·5 | ... | 15 |
| | 8 | 64 | |
| 1 | 8·5 | ... | 17 |
| | 9 | 81 | — |
| | | | 65 |

*i.e.* the sum of the terms shown in the $\delta y$ column 9 ... 17 is given by the difference between the 81 and the 16 of the $y$ column.

It is well to spend time upon these points until they are clearly understood, a good number of examples being worked and explained. It is then pointed out that it is not necessary to graph the integral curve in order to get the result, but that its equation may be written down and the lengths of the required ordinates calculated.

Thus, given that $\frac{\delta y}{\delta x}=2kx$, $\therefore$ equation of the integral curve is $y=kx^2$.

When $\quad x=1, \quad$ then $y_1=k$,

$$x=2, \qquad y_2=4k;$$

$$\therefore \text{ area}=y_2-y_1=3k,$$

*i.e.* the area contained by the line $\frac{\delta y}{\delta x}=2kx$, the $x$-axis and ordinates at $x=2$ and $x=1$.

A number of examples should now be worked, and the work should, I think, be set out in the form indicated above, and every now and again the actual curve should be drawn and the length of the line read off. In other cases the diagram to scale may be omitted and a small sketch of the figure drawn.

Then later on the usual method of setting out the integral sign may be shown, *e.g.* area$=\int_a^b y\,\delta x$. But it is very essential to emphasise that the line on the integral curve represents the area on the derived curve between the same ordinates.

Concurrently with the working of these exercises is a convenient time to deal with the cubic curve. Taking the type $y=kx^3$, I have begun with the curve $y=\frac{x^3}{6}$ by drawing up a table of coordinate values of $x$, $y$, $\delta x$, $\delta y$, and $\frac{\delta y}{\delta x}$.

| $\delta x$ | $\delta x_1$ | $x$ | $y$ | $\delta y_1$ / $\delta y_2$ | $\delta y$ | $\frac{\delta y}{\delta x}$ |
|---|---|---|---|---|---|---|
| | | ·5 | $\frac{1}{48}$ | | | |
| | ·5 | | | $\frac{7}{48}$ | | |
| 1 | | 1 | $\frac{1}{6}$ | | $\frac{13}{24}$ | $\frac{13}{24}$ |
| | ·5 | | | $\frac{19}{48}$ | | |
| | | 1·5 | $\frac{9}{16}$ | | | |
| | ·5 | | | $\frac{37}{48}$ | | |
| 1 | | 2 | $1\frac{1}{3}$ | | $2\frac{1}{24}$ | $2\frac{1}{24}$ |
| | ·5 | | | $1\frac{13}{48}$ | | |
| | | 2·5 | $2\frac{29}{48}$ | | | |

In this table the alternative values of $\delta y$ are distinguished as $\delta y_1$ and $\delta y_2$ respectively, the two values taken together being called $\delta y$, and denoting the increment of $y$ about its mean position.

The exercises that follow are:

(1) The curve $y=\frac{x^3}{6}$ is graphed by plotting $y$ against $x$.

(2) $\frac{\delta y}{\delta x}$ is plotted against $x$ and verified graphically, or the previous curve is differentiated.

(3) This curve is compared with the curve $y=\frac{x^2}{2}$, which has been drawn previously, testing it either by tracing paper or by measurement, and noting that it is the same curve, but is shifted slightly in position.

It is, in fact, raised $\frac{1}{24}''$, when $\delta x=1$.

It is then shown that if the interval $\delta x$ were taken larger than 1, then the derived curve would be raised still further; and again, if $\delta x$ be smaller than 1 the shifting of its position would be reduced.

We then proceed to investigate this by taking the general case

Let
$$y=kx^3;$$
then
$$y-\delta y_1=k(x-\delta x_1)^3,$$
and
$$y+\delta y_2=k(x+\delta x_1)^3;$$
$$\therefore\quad \frac{\delta y_1+\delta y_2}{2\delta x}=k\{3x^2+(\delta x_1)^2\}$$
or
$$\frac{\delta y}{\delta x}=k\left\{3x^2+\frac{(\delta x)^2}{4}\right\}.$$

This is verified numerically from the previous table, taking
$$k=\tfrac{1}{6}\text{ and }\delta x_1=\cdot 5,$$
and it is shown that in this case
$$k\{(3x^2)+(\delta x_1)^2\}=\frac{x^2}{2}+\frac{1}{24}.$$

Several cases are worked out, such as when $\delta x=2$, $\frac{1}{4}$, ·1, etc., and in this way it is shown that while in practical work it is not possible to eliminate entirely the error, still we can see that as $\delta x$ becomes smaller and smaller the result approximates to $3kx^2$, and that if we assumed that the effect due to the size of $\delta x$ were entirely eliminated, then the result would be $3kx^2$. We can therefore compensate for this error, and so obtain the true result.

In graphic work this can be easily done by drawing a line $Ox_1$ parallel to the $x$-axis at a distance $k(\delta x_1)^2$, and taking this instead of the true axis. In this way we get $y=3kx^2$ as the correct derived curve of $y=kx^3$.

Exercises follow to illustrate the above facts, in which the practice is adopted of writing $\frac{dy}{dx}$ when it is assumed that the error due to the finite size of $\delta x$ has been eliminated.

The two curves, $y=kx^3$ and $\frac{dy}{dx}=3kx^2$, are arranged on similar axes, and it is shown that an area on the derived curve is represented by the difference of the two corresponding ordinates on the integral curve, and this result is verified by counting squares, and by Simpson's rule.

Plenty of exercises follow on differentiation, integration, and finding areas.

Being now able to find the area of the figure contained by the straight line $y=a+bx$, the $x$-axis, and the ordinates at $x_1$ and $x_2$, we proceed to find the volumes of solids of revolution. We assume that the figure rotates about the $x$-axis, and that then the strip $\frac{\delta y}{\delta x}\times\delta x$ or $y\delta x$ becomes a narrow disc, of volume $\pi . y^2\delta x$; and the sum of all these discs into which the space between $x_1$ and $x_2$ is divided is the volume of the frustum of a cone, and may be represented as $\pi\int_{x_2}^{x_2} y^2 dx$, in which $y=a+bx$.

Next, if in the expression $V=\pi\int_{x_1}^{x_2} y^2 dx$ we write 1 instead of $\pi$, we obtain the volume of a square pyramid.

Also $\cdot 433\int_{x_1}^{x_2} y^2 dx$ will give the volume of a pyramid, with an equilateral triangle for base, and in this way the volumes of the regular pyramids are worked out.

Not infrequently the curves are graphed, and it is well to make the boys realise that the various cross sections of the pyramids are represented respectively by the ordinates of parabolas, while in the cubic curves, which are the integrals of the parabolas, the ordinates represent the volumes of the pyramids.

W. KNOWLES.

*(To be continued.)*

## REVIEWS.

(1) **Map Projections.** By A. R. HINKS, M.A. 5s. (Cambridge University Press.)

(2) **Maps and Survey.** By A. R. HINKS, M.A. 6s. (Cambridge University Press.)

(3) **The Text-Book of Topographical and Geographical Surveying.** By Col. C. F. CLOSE, C.M.G., R.E. 3s. 6d. (H.M. Stationery Office.)

(4) **Map Projections.** By J. L. CRAIG, F.R.S.E., Survey Department, Cairo.

(5) **The Theory of Map Projections.** By J. L. CRAIG, F.R.S.E., Survey Department, Cairo.

(6) **Maps and Map Making.** By E. A. REEVES, F.R.G.S. 8s. (Royal Geographical Society.)

While the above treatises are mainly designed to meet the wants of those engaged in various stages of training for survey work, we believe that they will be consulted and appreciated by a much wider public, and that they will be studied by many readers of the *Mathematical Gazette*. The keener interest now taken in Geography as a school subject, and the widened outlook of the teacher of Mathematics may well encourage this hope. We have heard much of the fusion of Mathematics and Science, and it must be admitted that the practical steps taken by the Committee of the Association to promote it have added greatly to the interest of the Annual

Meeting. The promoters seem to have had chiefly in view the fusion of *Physics* and Mathematics, but that of *Geography* and Mathematics is, in the interests of education, almost as much to be desired, and the "mathematical master" will do well to bear this in mind. He is now liable at any time to be consulted by a geographical colleague on various points of Mathematical Geography, and while some questions on this subject admit of fairly simple answers, others may be asked, especially on Maps and Map Making, which may give trouble.

Each of these masters may say that the scheme of work in the school allows very little time for the special teaching of the mathematics of geography, and, in the case of the geographical master, this is perhaps painfully true. But with cordial co-operation between the two something may be done. The less time at their disposal the more need there is for their teaching to be well planned. It is good that there should now be available such excellent works to consult as those in the above list. It has been difficult hitherto to obtain information on the subjects they treat which shall be at once full and authoritative; the sections in geographical text-books were sometimes too scrappy and too often inaccurate, though a great improvement has been noticeable of late.

(1) Mr. Hinks, we think, strikes quite the right note in his preface. He has not written a theory of map projections from a merely mathematical point of view, but with the object of giving clear accounts of the special merits and demerits of about thirty projections in use, and of enabling his readers to recognise what a map, constructed on one of these, will do and what it will not do. This plan certainly seems the best for the students for whom the work is designed, while it need cause little inconvenience to those who wish to work up the perspective projections (which here fall rather into the background), for they can readily find what they want in older treatises or deduce it from the principles of Projective Geometry. He passes at once to projection in the wider sense, enumerates the various desirable properties which a map should have, and proceeds to show which of these can be secured with more or less completeness; for the construction of a map, like that of a battleship, must, as a rule, be based on compromise.

Having briefly discussed the various properties which it may be advisable for a map to have, as regards orthomorphism, areal equivalence, and orientation, the author devotes successive chapters to the principal *systems* of map projection under the headings respectively of (i) Conical, (ii) Cylindrical, (iii) Zenithal, (iv) Modified Conical, treating (ii) and (iii) as limiting cases of (i), pointing out how either orthomorphism or areal equivalence may be obtained, but deferring the mathematical proofs to a later chapter. Those zenithal projections which are perspective have a separate chapter to themselves. The gnomonic projection of the sphere on the six faces of an enveloping cube is treated with some detail on account of its importance in finding great circle courses between distant places. A useful reference is here given to a paper by Prof. Turner in the "Monthly Notes" of the *Royal Astronomical Society*, vol. lxx. p. 204, entitled: "On the Diagrammatic Representation of Proper Motions." De Morgan wrote a small treatise in explanation of this projection, but relied for finding required points on numerical calculation where Prof. Turner gives geometrical constructions.

Of the chapters on these various projections, the most generally interesting are II. (on Conical Projections), and III. (on Modified Conical and Conventional Projections), and of these the sections on the Polyconic and its applications to the International Map on the scale 1 : 1,000,000 are perhaps the most attractive. Mollweide's (now making its appearance on the walls of our school rooms) has its merits and defects tersely put. The account of the mathematical principle underlying it may be quoted as a fair example of Mr. Hinks' style:

"Consider first the construction of a hemisphere on this projection. Take $r=\sqrt{2}R$, where $R$ is the earth's radius. Then the circle with radius $r$ has the same area as the hemisphere.

"Take a diameter of this circle to represent the half of the equator. and divide it equally into, say, six parts, each representing 30° of longitude. Draw ellipses through the poles and these points on the equator to represent the meridians. By an elementary property of the ellipse the areas of these gores are all equal. The parallels of latitude are straight lines parallel to the equator. The distance from the equator of a parallel of latitude $\phi$ is $r \sin\theta$, where $\pi \sin\phi = 2\theta + \sin 2\theta$. This equation cannot be solved directly to find $\theta$ where $\phi$ is given, but by the

reverse process it is easy to find $\phi$ for any given value of $\theta$, and when this has been done for a sufficient number of cases, the values of $\theta$ corresponding to any desired value of $\phi$ can be interpolated."

A table giving the values of $\sin\theta$ to three decimal places at intervals of 10° is added. Previous possession of this table would have saved us a considerable batch of numerical calculations, for, wishing to test a published map, or rather our own ideas by means of it, we had made a table of our own. We do not, however, regret this, as it has enabled us to check the accuracy of Mr. Hinks' table for the values of $\theta$ corresponding to latitudes 10°, 20° and 40°, and we have found complete agreement.

There is an interesting reference to the *transverse* Mollweide. A reproduction of the corresponding map by Col. Close, forms an attractive frontispiece to the book.

Among some other little known projections is given an interesting one by Aitoff.

It is thus described:

"Take the zenithal equal area projection of a hemisphere and pass through the straight line representing the equator a plane making an angle of 60° with the plane of projection. Project the net of the zenithal projection orthogonally on this plane. We then have a projection of the hemisphere, bounded by an ellipse of which the major axis is twice the minor. Now, halve the scale in longitude; that is to say, number the meridian which was 10° from the central meridian 20° and so on. We thus obtain a representation of the whole sphere within the boundaries of the ellipse."

Chapter VII. gives an account of the various projections in topographical maps and atlases; Chapter VIII. of the simple mathematics of projections. Here we have a simple treatment of the various special cases, with a reference to Germain for the general theory. Amounts of error in radial distance, azimuth, etc., are tabulated in Chapter IX. A set of tables facilitating construction is added.

(2) In *Maps and Survey* Mr. Hinks shows the same power of lucid exposition as in *Map Projections*. Having discussed in Chapter I. the topographical information that it may be desirable to have given in maps, the conventional signs adopted for giving it, and the improvements in course of progress for making these still more effective, he proceeds to a detailed analysis of about thirty of the more notable productions of the survey departments of our own and other countries, pointing out the merits and defects of plan and execution. The question which naturally arises to English readers—*How do British maps compare with those of other European countries?* he answers thus: "*There are isolated examples of topographical maps, new series of which only a few sheets have appeared, which are in some respects better than the British maps. . . . But on the other hand, there is no country which has maps of greater variety and completeness, and of more uniform excellence.*" A very interesting section of this chapter deals with the International Map of the World on the scale of 1 : 1,000,000, now being made in accordance with the unanimous decision of delegates from the principal Governments of the World at the Congress held in London in 1909. Samples of this are reproduced representing widely different tracts—the southern littoral of Turkey and the plateau of Griqualand. Chapters III., IV. and V. deal respectively with Route Traversing, Simple Land Survey, and Compass and Plane Table Sketching. A clear and interesting account is given of the method of the Triangle of Errors, through Capt. Robinson's solution of which the Plane Table has been improved into an instrument of precision.

Chapter VI. gives a detailed account of the various objects at which any topographical survey worthy of the name should aim, and of the methods and instruments used for attaining them. It contains in addition an interesting historical sketch of the Ordnance Survey of Great Britain and Ireland, and an account of the surveys now in progress in various parts of the Empire.

Chapters VII. and VIII. deal with Geodetic Survey and Survey Instruments.

The work is illustrated by twenty-four plates. Nine of these are devoted to specimen reproductions of maps; the rest show instruments and processes of survey. Of the former, one of the most effective is that of Basutoland, showing the ability of contours to indicate relief; of the latter, we may notice that showing operations for the determination of the Uganda-Congo boundary.

(3) Major Close's book deals in one volume with the same two subjects as (1) and (2). It is of a more technical character, going into minute details of survey work, especially in the case of astronomical observations. Besides chapters on "Survey under Active Service conditions," "Levelling," "Map Reproduction," "Field Astronomy" and other subjects, it contains a section on "Occultations" (Capt. C. H. Foulkes), showing how to find Longitude through the occultations of a star or planet by the moon, and illustrated by two large folding diagrams of Col. S. C. N. Grant for graphic computation. To the unprofessional reader, and especially to the map-lover, the most interesting features of the book are specimens of graticules of various projections and reproductions of maps published by various survey departments. Among the most effective of the latter are Bizerta (French 1 : 50,000), St. Helena (2·5 in. to the mile) and a U.S.A. district. There is also a set of six star maps, mounted on canvas, showing all N.A. stars down to 4·5 magnitude. It is not often that there is a chance of making such a valuable addition to one's mathematical library at such a low price. It does not appear to be nearly so well known as it deserves to be.

(4) This is the reprint of a technical lecture, delivered January, 1909, dealing with the subject rather from the point of view of the geographer, who requires simply a working knowledge of general principles, than from that of map maker, who has to select suitable projections for his purpose, or from that of the mathematician concerned with the most general theory of map projection. The exposition is clear and well illustrated by numerous diagrams. The use of the indicatrix in showing errors of scale, change of area, and transformation of angles is explained. About twelve of the most important projections are described, and large diagrams given illustrating the Sanson-Flamstead, the Equivalent Modified Conical, Gauss' Conformal, and the Equivalent Zenithal.

The one to which the reader will turn with the greatest interest is the "Mecca Azimuthal Projection." The general problem of such "retro-azimuthal" maps does not appear to have been fully investigated as yet. The graticule for the one here adopted, merely with the purpose of giving the proper azimuth for Mecca for all places marked on it, has a simple method of construction. The meridians are represented by a set of equidistant parallel straight lines. Any point where a definite parallel of latitude cuts one of these meridians is found by calculating the azimuth of Mecca at the point chosen, and then drawing a straight line from Mecca on the map, making an angle with the meridian equal to this azimuth to meet the meridian of the chosen point. A sufficient number of points on a parallel of latitude having been thus found, a parallel is drawn freehand through them. Straight lines through Mecca on the map may easily be seen not to be projections of a loxodrome or rhumb-line, though they cut all the *projected* meridians at a constant angle. We learn from Major Craig's second pamphlet that each of the corresponding lines on the surface of the sphere is its intersection with a hyperboloid of one sheet.

A valuable bibliographical table is added.

(5) After a general introduction to the theory special attention is given to the projections actually used in the Egyptian Survey Department. The earth being treated as a spheroid of revolution, the system of curvilinear co-ordinates due to Gauss is explained. We have a chapter mainly on the Indicatrix and its application to questions of distortion. This is followed by one, which, after general questions of orthotomic section, conformality, rectilinearity of meridians or parallels, treats of several projections. The Gauss Conformal Projection has a chapter to itself. There is an appendix of tables, and a frontispiece showing the Sanson-Flamstead projection on a scale 1 : 2,000,000, and illustrating the use of the Indicatrix.

(6) Mr. Reeves has done both students and teachers of geography a great service in publishing these three lectures, delivered in 1909 to the Royal Geographical Society. The lantern slides shown when the lectures were given are here replaced by illustrations, diagrams, and maps. In the first a sketch is given of the history and development of the more important surveying instruments. We have descriptions of the scaph, the back-staff, the cross-staff, and the cross-bow, instruments now rarely met with, except in museums, though we have a friend who has secured within recent years a Davis' back-staff and an astrolable from curio shops. Possibly some hearers or readers of these lectures may now be able to recognise some such

instruments still among the lumber of country dealers. The second lecture is on survey methods, and is also in part historical. We have diagrams of charts of the Atlantic (1565), of the Mediterranean (1500), and of Africa, of the triangulation of the British Islands and of India, as well as pictures of survey work as now actually carried on. The third deals with cartography. A few of the more important projections are described, distortion in form being illustrated by projections of the same gigantic human face on the stereographic, orthographic, globular, and Mercator's projection. This use of the human face as a sort of indicatrix, though designed originally for a lecture to boys, will be welcomed by older readers, especially by such as have to teach geography. The processes of map construction and of printing and their difficulties are explained, and we learn that a large coloured map, such as we are used to seeing in the *Geographical Journal*, may cost anything from £25 to £60 to bring out. The book should be in the library of every secondary school.

EDWARD M. LANGLEY.

**Matrices and Determinoids.** C. E. CULLIS. Pp. 430. 21s. net. 1913. (Cambridge University Press.)

This handsome volume "deals with *rectangular matrices* and *determinoids*, as distinguished from *square matrices* and *determinants*, the determinoid of a [non-quadrate] rectangular matrix being related to it in the same way as a determinant is related to a square matrix." The idea of instituting such an analogue to a determinant must have occurred to many students of mathematics. Probably the least well-advised of them was Dostor, who made the innovation of calling

$$\left\|\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right\|$$

a "determinant multiple," and of defining it as the equivalent of

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Although the "determinoid" of Professor Cullis

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

has the same meaning, it is not at all because he proceeds in the same way: he has too much insight and logical acumen to adapt such a course. He defines a "determinoid" quite independently of the conception of a determinant, namely, as an aggregate of products of elements of an array, and he gives a rule for the formation of the products and a rule of signs: he then proceeds to elaborate the consequences of his definition. As a determinant is a special case of a determinoid, it is clear that this definition must include and degenerate into the usual definition of the former.

The book contains in all eleven chapters. The first (pp. 1-21) is generally introductory, and deals in particular with definitions and notations connected with the new entities; the second, third, and fourth, extending in all to as many as eighty-two pages, deal with questions of signs; the fifth (pp. 105-152) concerns the various expansions of a determinoid; the sixth (pp. 153-208), a valuable chapter, treats of matrices in the special Cayleyan sense,—the equality, addition, subtraction and multiplication of them; the seventh (pp. 209-247) deals with the determinoid of the products considered in the sixth; the eighth concerns matrices whose elements are minor determinants of a fundamental matrix; the ninth and tenth,—valuable chapters quite independently of determinoids,—concern the rank of a matrix and the solution of matrix equations of the first degree; and the eleventh (pp. 364-417) contains a full and fresh treatment of the solution of a system of ordinary linear equations, the system being viewed as a matrix equation of the first degree.

In form the exposition is elaborately logical. It also bristles with words and phrases used in unfamiliar ways; all of them, however, are carefully defined, and there is, in addition, a helpful Index, twenty-two columns in extent, which can be used as a Glossary.

Unlike the old text-books on determinants this on determinoids has only one chapter that can be viewed as treating of an "application," namely, the last.

The author promises, however, to follow up his present work with two other volumes, the greater part of which would be occupied in remedying this seeming defect, the subjects to be dealt with being ordinary algebra, the algebra of quantics, algebraic geometry, and vector analysis. The appearance of these additional volumes will be looked forward to with much interest, as they can scarcely fail to be of considerable service in the advancement of science. It will also then be apparent to what extent the theory of determinoids is likely to contribute towards such advancement. So far as one can at present see, there is safety in hazarding the conjecture that the extent will be limited as compared with that of its fellow-subject Matrices, or its prototype Determinants. THOMAS MUIR.

**Problem Papers. Supplementary to Algebra for Secondary Schools.** By C. DAVISON. 8d. 1913. (Cambridge University Press.)

This pamphlet contains 160 selected questions in Algebra, with answers, from papers set, presumably by the author, for the Intermediate Board, Ireland.

The questions cover the ordinary range of algebra up to progressions and theory of quadratics. They are clearly put and in good style, and the collection forms a useful and inexpensive supplement to the contents of the standard text-books.

C. S. J.

**The School Algebra** (Matriculation Edition). By A. G. CRACKNELL, M.A., B.Sc. Pp. 420 + lxv. Price 4s. 6d. (Clive & Co.)

As a school text-book, this volume has much to recommend it, the examples and test-papers being carefully chosen and very full. The idea fundamental in the advice "whenever the student finds any difficulty in expressing the work algebraically, he is advised to invent for himself a corresponding question in arithmetic with some simple numbers" is the main-spring of the book.

Regarded as an exposition of Algebra the volume is scarcely so happy. To begin with, the author considers one of the main difficulties in teaching Elementary Algebra to be the multiplicity of the rules. Surely this is the fault of the text-books, not of Algebra. One almost suspects that the author does not realize what Algebra is. He defines Algebra as a "kind of advanced arithmetic in which numbers may be represented by letters." Can this definition be accepted?

The theory of so-called Arithmetical Fractions (Art. 16) is Algebra; for the symbol $\frac{2}{3}$ is quite as "imaginary," as an abstract number, as $\sqrt{-1}$.

The author rightly assumes the Commutative Law for Multiplication and Division in Arithmetic and its extension to fractional forms, but incorrectly assumes, in Art. 68, that it is true for fractions in general. The only correct method is to *define* multiplication and division in general for fractions in such a way that the laws already assumed (or proved quite easily) for fractional forms are not contradicted. The definition of the new meaning of the $-$ sign as a *reversal* is far too sweeping: it is not logically necessary that, if $+(-5)$ is defined as reverse of $-5$, that it should follow that $A \times (-5) = -(A \times 5)$, so that Arts. 44, 45 hardly prove what they profess to do. The author states: "No satisfactory proof can be given of the law of signs for multiplication when the signs indicate positive and negative quantities—at any rate no proof which will really apply to all possible cases." The meaning of this is not quite clear, but if it refers to the proof of the rule $(\pm a) \times (-b) = \mp ab$, it is incorrect. There is no proof at all for this "rule": in fact it is not a rule, it is the *definition* of a negative multiplier.

As this is a volume for matriculation students, the whole treatment of negative quantities and zero in Chapter XIV. is unfortunate; for it is getting quite usual for little bits of theory such as "Prove (illustrate or explain) that $2-(-3)=5$," to appear in matriculation papers. This cannot be proved, although the author apparently professes to do so in general. It can only be *defined* as the meaning to be attached to subtracting a negative so that the laws of Arithmetic are not contradicted, or *illustrated*, by reference to a thermometer, debit and credit, etc. It is also unfortunate that the author perpetuates the cause of much muddle amongst beginners in Calculus by suggesting that "0" or "zero" ever stands for anything else but "absolute nothingness." The same remark applies to the use of the word quantity for number and calling infinity a quantity. It cannot be too strongly impressed that $(x^2-a^2)/(x-a)$ is meaningless when $x$ is absolutely equal to $a$, though a meaning can be found for all other values of $x$ no matter how small $x-a$ is.

The chapter on Fractions contains "proofs" of the Laws "as in Arithmetic": these, depending on the idea of dividing the unit, are unsound as laws of abstract number. They apply only to "fractions in measurement," concrete quantities.

There are one or two little slips, such as in the definition of "term," Art. 27. The terms of $2a+b-3cd$ are $(2a)$, $(b)$ and $(-3cd)$, and not $2a$, $b$, $3cd$ as stated.

It is not stated that the multiplicand can be distributed as well as the multiplier, and yet in Ex. 12, Art. 37, we have both operations performed. It is not *necessary* to take out inner brackets first; it is in general convenient, but a boy in the sixth form would probably remove them all simultaneously. The note at the foot of page 73 should be deleted: whatever $a$ is (including infinities) $3a+5a-8a=0$.

The treatment of Graphs is one of the best features of the book: the chapter on the use of formulae is also an innovation deserving all commendation; ratio and proportion receive adequate treatment; but surds suffer from a lack of preparation as to approximate values. X. Y. Z.

**A School Arithmetic.** By A. Clement Jones, M.A., Ph.D., and P. H. Wykes, M.A. Pp. 440+xxxii Answers. Price 4s. 6d. (Arnold.)

This is a very useful text-book, especially noteworthy for the excellence of its revision exercises and miscellaneous exercises and problems. Everything is beautifully explained, and the authors have not hesitated to use literal symbols to generalize the rules. The development of fractions on the idea of subdividing the unit, although I do not care for it, is perhaps not so unsound in an Arithmetic as it would be in an Algebra, especially as the authors are careful not to profess to prove the rules, but only to justify them. Still I should like to see the rules proved for fractional forms and then extended. I am sure it could be done in an Arithmetic quite simply without wearying or frightening the beginner, and yet perfectly strictly. It is a treat to see Interest, etc., considered as simply variations of the one great principle of proportion, coupled with a satisfactory account of the Unitary Method. Approximate Methods for Decimals and Logarithms are well done. There is also a large section on Mensuration, which should make the work interesting. A very good book.

**A General Course of Pure Mathematics.** By Prof. A. L. Bowley, Sc.D. Pp. 268. Price 7s. 6d. net. 1913. (Clarendon Press.)

Perhaps the only adverse criticism that can be made against this excellent volume is that one is not clear as to the section of the public for which it is intended. It is admittedly not for the mathematical specialist; for the engineering student, it is, more's the pity, liable to be considered too rigorous, preference being given to one or other of the advanced treatises labelled "Advanced Practical Mathematics"; and, apparently, in England these two classes alone are worth consideration!

The volume starts from an assumed knowledge of "Matriculation Mathematics" and extends to and slightly beyond "Scholarship Mathematics." The proofs are beautifully clear and rigorous, and worked out on up-to-date lines. The opening article on the rule of signs is worth the attention of many an author of elementary mathematical text-books. We also find on p. 3 a cross-reference to the possibility of finding a value for an incommensurable index, *e.g.* a logarithm, without which the proofs of the laws for logarithms are worthless; the idea is assisted by obtaining $\log_{10}2$, $\log_{10}3$ from the calculation of $2^n$, $3^n$ in a highly suggestive manner. Variation is illustrated graphically in such a way as is likely to convince even a dull student. Limits are carefully treated, the definition being given accurately, though in a slightly unusual form, the essential point that $|f(x)-l|$ becomes and *remains* less than $\epsilon$ being excellently brought out. The author's definition is, however, in my opinion, inferior to that derived from an endless sequence, as given by Hobson, and the whole matter might be improved by a preliminary, graphical, and arithmetical treatment of "large" and "small" numbers. The usual present-day symbol **lim** would look better than the author's sign for limit. The proof of the Logarithmic Series does not seem so satisfactory as some of the other proofs; the author is rearranging a double series without having given any preliminary work on the permissibility of this. Complexes are treated from the standpoint of an operator, and worked to a logical conclusion: but my experience has been that no student except the mathematical specialist ever gets a grasp of an

operator; on the other hand, the introduction of complexes as a doubly-ordinal class of numbers leads to vectors and two-dimensional notation without difficulty, and the connection with ordinary Algebra is evident.

After finishing the perusal of such a book as this, we feel what a pity it is that, leaving higher work out of the question, we fail to convince the non-mathematical student that he requires a knowledge of the theoretical side, even of Matriculation Mathematics.

**A Shorter Algebra.** By W. M. Baker, M.A., and A. A. Bourne, M.A. Pp. 320+lvii Answers. Price 2s. 6d. 1913. (G. Bell & Sons.)

This volume contains most of Part I. and about half of Part II. of the author's *Elementary Algebra*; it has been published as a volume "thoroughly adapted to the syllabus of such examinations" as the London Matriculation, Oxford and Cambridge Locals, Preliminary and Junior, etc.

As such, it retains all the good points—and all the bad ones—of the authors' more extended work. The examples and revision papers are excellent, and there is a useful selection of specimen examination papers at the end of the book. As to the bad points, those of theory, such as the page on Negative "quantities," and the proof of the rule of signs, in which it is assumed that because $-2$ has been defined as the reverse of $+2$ *as a number* it bears the same interpretation *as an operator*, it is unnecessary to say anything here—the reader is referred to No. 42 of the Special Reports of papers prepared for the International Commission on Mathematics recently held at Cambridge. There is the usual mistake of stating that $a^0 \equiv 1$ and $a^{-n} \equiv \frac{1}{a^n}$ for *all* values of $a$; no exception is made for $a=0$, although in this case division by $a^n$ is meaningless. The "rule" $\sqrt{a}\,.\sqrt{b}=\sqrt{ab}$ seems to be assumed, perhaps wisely, without any attempt at proof. Ex. 2, p. 248, is unfortunate in suggesting, by reference to the reason previously given for "rationalising the denominator," namely the saving of time in arithmetical evaluation, that $1/(2+\sqrt{5}-\sqrt{3})$ is a case in point. No student would try to "save time" in such a case as this by rationalising the denominator, unless he was one of those unfortunates who have never been taught approximate multiplication, division and square-root processes. The usual "proof" of the *definition* of equality of $a+\sqrt{b}$ and $c+\sqrt{d}$ is given.

In spite of these faults, however, the book, in the hands of a careful teacher, should form an excellent class-book.

**Bell's Outdoor and Indoor Experimental Arithmetics.** By H. H. Goodacre, E. F. Holmes, C. F. Noble, and P. Steer. In five parts. 3d. each part. 4d. limp cloth. 1913. (G. Bell & Sons.)

The object of this series is to lead children—by experiment, inference, comparison and contrast—to form a real conception of Length, Area, Weight and Volume. This aim is kept in view, and is well carried out. Judging by the books, it would seem that this is the result, not of mere theorising, but of a definite scheme that has already been proved successful in practical experience.

**Practical Mathematics.** By Norman W. M'Lachlan, B.Sc., A.M.I.E.E. Pp. 180. Price 2s. 6d. net. 1913. (Longmans.)

This book rather deserves the name of "Mensuration for Engineers." It consists of proofs of all the formulae—with the one exception of the truncated-cylinder volume—applying to the plane and solid figures usually met with in practice; added to this there is a very large number of worked-out examples, together with a fine collection for the student's work. The style of these worked-out examples is excellent, and should prove useful in correcting the usual slovenly working which the engineering student seems to think "will do." As a set of examples for the pure mathematician who has to teach Practical Mathematics, the book meets a badly-felt want.

**Plane Geometry.** By W. B. Ford and C. Ammerman. Pp. 213+tables. Price 3s. 6d. net. 1913. (Macmillan Co., N.Y.)

This is an attempt to provide a rational course of Geometry intermediate in style between the Euclid of our youth and the ultra-modern substitute. The volume opens with the construction of simple figures, and a good section on funda-

mental ideas, otherwise definitions unlabelled. Then follow deductive sections on Rectilinear Figures, the Circle, Proportion and Similarity, Areas, etc., and an appendix on Maxima and Minima. As a book for ordinary class-work with beginners, the proofs of the theorems suffer in many cases from the "talky" style adopted, while there is no attempt to accentuate the importance of the theorems by a difference of types or size of type. To English readers some of the enunciations appear strange; *e.g.* "An inscribed angle is measured by one-half the intercepted arc"—a form no doubt adopted from French text-books. The section on Maxima and Minima is very good, though short. For the rest, this is certainly one of the most interesting and suggestive school-texts I have ever seen. But there seems a tendency now to hark back to the so-called "logical treatises" of thirty years ago. One cannot help noticing the continual plaint of examiners against the slovenly style in which theorems and riders are written out nowadays. J. M. CHILD.

**The Theory and Practice of Mechanics.** By S. E. SLOCUM. Pp. xlii + 442. 15s. net. 1913. (Constable.)

Professor Slocum's volume represents the endeavour to find a *via media* between the course in theoretical mechanics in which theory is completely divorced from practice, and the course of applied mechanics in which the range is apt to become "so technical and limited as to obscure the meaning and scope of the fundamental principles involved." With that aim in view he endeavours to arouse interest in the fundamental principles by connecting them, not once or now and again, but all the time, with problems of a practical nature such as are already within the experience of the students for whom the book is in the first place intended. On the other hand, he is one of those who would feel happier were he assured that this Course would not be sold by the owner immediately he has passed his examination, or laid aside never to be referred to again. He thinks that if more is contained in the College text-book than is needed during the period of probation, the student will be more ready to fly to the familiar volume for further information than to a book of reference with which he is entirely unacquainted. Hence the presence in this volume of much that is beyond the immediate needs of those taking an elementary course. He hopes, indeed, that by those unwonted inclusions the work will be available for the purposes of students in mechanical, civil, and electrical engineering. He further suggests that the teacher will be aided in the attempt to get rid of the old objectionable water-tight compartments by studying carefully his arrangement of material. "For instance, the subject of elastic vibrations . . . involves a physical interpretation of Maclaurin's theorem, simple trigonometrical relations, the graphical representation of trigonometrical functions, elementary differential equations, the definition and properties of harmonic motion, the composition of waves, the formation of nodes in vibrating strings, and practical engineering applications, although the discussion is so simple as to be easily intelligible." The book opens with some 40 pages of the usual tables of constants. Individual opinions no doubt will vary as to this being an appropriate place for tables. Personally we have a weakness for the 40 sheets in a pocket in the cover, or quite separate in thin pamphlet form. Sixty pages are then devoted to the geometry of motion "based on the intuitive ideas of time and space"—the methods of the calculus are used, by the way, from the outset. "Kinematics" closes with sections on the instantaneous centre and centrode, piston speed curves, velocity ratios in mechanisms, sectorial velocities, and Kepler's Laws. The next section of some fifty pages is given up to the treatment of fundamenfal principles. The student is shown how the fundamental equation $F=ma$, with the equations of work, energy, impulse and momentum express only what is given in Newton's laws, and then by a natural extension are found the more general relations—the principle of the conservation of energy, d'Alembert's principle, and the principles of virtual and of least work." But he is warned that Newton's axioms are not the only possible logical foundations, and is duly referred to Hertz and Mach. The chapter headed "Statics" discusses problems connected with the centre of gravity—there is apparently no mention of Simpson's rule—and passes on to the finding of areas and volumes, to forces and their equilibrium, to structures and flexible cords. Forty pages on friction and lubrication bring us to The Kinetics of Particles, and this in turn to about 80 pp. on The Kinetics of

Rigid Bodies. The determination of moments of inertia of the standard sections is fully dealt with, and the inertia and gyration ellipsoids are arrived at, with the reciprocity of the Poinsot solids. Lagrange's equations are obtained and illustrated by various problems, including that of the bell and clapper—some of our readers will remember the experiments performed with the Cologne Kaiserglocke. We now come to statically indeterminate problems and the Principle of Least Work, with seven or eight fully worked out examples. Hamilton's Principle is applied to transverse vibrations of beams and rods, and several useful pages are given to the theory of models. This chapter closes with a discussion of the analogy between the equations of Dynamics and Electro-dynamics. Finally we have a chapter on the Dynamics of Rotation. Sections deal with Flywheels and Governors, Klein's graphical construction of the piston acceleration curve is explained, the student is introduced to indicator cards and polar diagrams, the mysteries of balancing are probed, and sections on gyroscopic action, the top, Brennan's monorail, the automatic torpedo invented by Admiral Howell, Griffin's grinding mill, etc., bring the book to a close. Professor Slocum sets forth his material with simplicity and directness, and his book is clearly the work of an experienced teacher. Teachers and examiners will find it well worth their while to seek the opportunity of perusing it, if only for the collection of over 400 examples (for the most part original). The publishers are to be congratulated on its production. Paper and type are good, the letters being large enough to inflict no strain on the eyes, the lines are generously spaced, and there is an abundance of well-drawn diagrams. There are notably few misprints. We recall one, in the value of the Moment of Inertia of the truncated cone.

**An Introduction to the Algebra of Quantics.** By Edwin Bailey Elliott, M.A., F.R.S. Pp. xvi + 416. 2nd edition. 15s. net. 1913. (Clarendon Press.)

It is a pleasant task to congratulate Prof. Elliott on the appearance of a second edition of his well-known book on the *Algebra of Quantics*. The first edition might well serve as a model of style for mathematical writing, and it has given the subject a place in mathematical examinations which it would hardly have obtained, unless the candidate had been able to get the necessary instruction so easily. It is true that the very perfection of the book was not without its practical drawbacks from an examination point of view, and pity may be spared for some of the victims. Did a teacher try to lecture on the subject? Anything worth saying was already in the book! Did an examiner congratulate himself on having invented an interesting and original problem? His colleagues would promptly refer him to the page and line! Did a candidate endeavour to "get up" the subject? The very facility with which he followed the reasoning militated against retention in the memory! The reviewer writes feelingly as a sufferer, but with the pleasant assurance that Prof. Elliott would be quite indifferent to any little drawbacks of this nature, when he had succeeded in his effort after a high educational standard.

In this new edition the Press has effected a considerable economy of space, partly by such alterations as

$$q^{-1}(l+mx/y) \text{ for } \frac{1}{q}\left(l+m\frac{x}{y}\right), \text{ or even } \partial^2 y/\partial x^2 \text{ for } \frac{d^2y}{dx^2},$$

and partly by closer printing Room has thus been found for much extra material, even though the number of pages is reduced; but it seems doubtful whether it would not have been better to submit to an increase of bulk. It must be confessed that the difficulty of reading is appreciably increased by the changes.

Prof. Elliott has wisely confined himself to verbal alterations throughout the majority of his treatise, the only noticeable change being the substitution of $\partial$ for $d$ in partial derivatives; while about a quarter of the book, or less, has been rewritten. A very brief introduction to the theory of symbolic methods and apolarity (§§ 60, 61, 207) is given, with references to Grace and Young's *Algebra of Invariants*. In § 157 Hilbert's second proof of Gordan's theorem has been added. An interesting section (§ 312) on annihilators of invariants has been inserted at the end of the book. Chapter V., on "Invariants as Functions of Differences," has been extensively altered, and so has that portion of Chapter XV. which relates to Boolian Complete Systems. In the first edition, § 229 (on

the reduction of the ternary cubic to canonical form) was unsound. In fact, the proof there given would (if valid) have established the fact that

$$x^3+y^3+z^3+6mxyz=0$$

could be a cuspidal cubic! This mistake is now corrected. The canonical reduction of a binary $(2n-1)$-ic in § 203 has been simplified with advantage. Perhaps § 204 might have been recast more thoroughly to correspond. The statement in § 224 that "The reduction of the sextic to the form $x^6+y^6+z^6+\lambda x^2y^2z^2$ has not, as a matter of fact, been effected"; while correct in the first edition, is untrue in the second (see *Messenger of Mathematics*, vol. xliii. 1913, p. 25). It was rather bad luck to be out of date so soon! The connexion of Faà de Bruno's theorem (§ 16) with § 7, Ex. 1, might have been pointed out with advantage, and also its extension to the case of quantics other than binary. Another, and perhaps an easier, proof is given in the examples to Chapter XIV. of Burnside and Panton's *Theory of Equations.*

One sentence in the preface will suggest rather sad reflections to the author's many friends. Tact and sympathy with the young student, self-sacrificing work for the cause alike of teaching and administration, and an example of untiring devotion to a high ideal are assets of the Oxford Mathematical School, the loss of which can hardly be contemplated without painful emotion.

**Hydraulics and its Applications.** With over 300 Figures and Tables. By A. H. Gibson. Pp. xv+813. 15s. net. 1912. (Constable.)

There are many text-books on the subject of Hydraulics and, as is natural, they are of varying merit. As a rule, it is well that the student who attacks a volume as comprehensive in its scope as that before us shall have had some preliminary acquaintance with elementary principles, and to that rule Prof. Gibson's book seems to be no exception. It will be an exceptional beginner who can make his way through its pages with reasonable rapidity and without considerable external assistance, and it is almost needless to add that the reader must in any case be familiar with the processes of the calculus. The intrinsic difficulties besetting the initiate arise from the fact that the theoretical treatment of the subject provides the student merely with the "rational framework on which to erect the more complete structure." Purely rational hydrodynamics is concerned with fluids that are neither elastic nor viscous, and whose particles when in motion preserve parallel paths. It is not surprising, therefore, that the student should find that his purely theoretical results turn out to be "flagrantly at variance," as Unwin puts it, "with the action of actual fluids." He sees that he must look upon results as mere approximations, and that in determining the values to be given to the empirical constants of his formulae we must take into account a host of data, the relevance of which is sometimes not immediately obvious at first. The teacher with ripe experience is fully aware that the key to complete comprehension is given to those only in whom the gift of interpretation is being developed as he passes from stage to stage of his training. It is not sufficient that a knowledge should be acquired of the results of experimental research. That is less than half the battle. It seems to us that one of the special qualities of Prof. Gibson's volume is that it discloses his possession of that faculty of knowing just when and where to give the suggestion, or the hint of a suggestion, which will induce the reader to reflect, and to consider why he must not follow up the apparently obvious clue, and if there are not some other neglected factors in play which have contributed to the final result.

A second edition of his text-book has been called for after the lapse of four or five years, during which experiment in Hydraulics has been incessant and productive of fruitful results. In such research the author himself has taken part, and has published important results since the book left his hands for the press. For instance, readers of the chapter on the resistance to the flow of air through pipes must not fail to consult *Engineering* for November 22 of last year. There they will find a discussion of the formula $dp=\dfrac{flv^2}{2gm}$ in the light of experimental results obtained by Messrs. Gibson and Grindley. From this it appears that such formulae only apply if $f$ is varied with the roughness of the inner surface of the pipe, the diameter of the pipe, the mean velocity of flow, the mean pressure, and the temperature of the air. We are glad to see that he has managed to get in the

results given in the *Proc. R.S. Edin.* 1912-13, with reference to the loss of energy consequent on oblique impact of two confined streams. His work on Water Hammer is also well known, and as here set forth will, like many another set of sections in the book, be of interest to the practical engineer as well as to the student. The amount of new material in this edition is so considerable that, taking into account the general revision, the rearrangements and the excisions, the volume may almost be said to be a new book. New sections have been added on waves of transmission and oscillation, on ripples, on the flow of fluids other than water, on the admixture of adjacent layers of fluid in pipes, and on the suction effect between passing ships. The application of Hydraulics to the design of Machinery is handled in nine chapters, covering more than half the book. The whole is now well up to date, and may be relied upon as a thoroughly adequate treatment of the subject. The book is beautifully printed and lavishly illustrated.

**Mechanics of Particles and Rigid Bodies.** By T. PRESCOTT. Pp. viii + 535. 12s. 6d. net. 1913. (Longmans, Green.)

Mr. Prescott's course of Applied Mechanics is intended to cover all that is needed for the pass degree at any of our Universities, with the exception of hydrostatics. It seems to be admirably adapted for that purpose. The author has considerable expository skill, and by the carefully selected examples which are worked out in every chapter he has given his volume the " utilitarian bias," which he hopes will make it useful to the budding engineer, and " interesting and live " to the science student. He anticipates the difficulties which beset the beginner when he for the first time has to turn his formulae into numbers, and he has gone far to smooth his path in this respect. The book contains a large selection of questions selected from various University examination papers and other sources, and, what will strike many readers as a valuable feature is the fact that in all cases the answers are given. Mr. Prescott draws attention to a question in which he is at variance with engineers, who, assuming maximum friction in two perpendicular directions at the same time, find a formula to which the author takes exception. The question is as follows, and it may interest some of our readers to look into the point at issue : "A friction cone is a frustum of a cone which fits inside a hollow cone and is turned by the friction between the inner surface of the hollow cone and the outer surface of the frustum. If the larger and smaller radii of the frustum are $b$ and $a$, the semi-angle of the cones $\alpha$, and the force pushing the frustum into the hollow cone $P$, what is the moment of the maximum friction about the axis, assuming the pressure to be uniformly distributed over the surfaces ? $\left[\frac{2}{3}\mu P \operatorname{cosec} \alpha \,.\, \frac{a^2+ab+b^2}{a+b}\right]$.

We have noticed a few misprints or slips, but they are not likely to give the student more than a moment's pause ; *e.g.* p. 116, l. 12 up, the sudden appearance of the unexpected $O$ ; l. 10, p. 120, insert " produced " after "$AB$" ; "$a^2$" for "$p^2$" in ($b$), p. 147 ; "$y$" for "$\gamma$" in l. 2, p. 167, and so on. The teacher will be well advised to consider the many merits of Mr. Prescott's volume, and in our opinion it certainly deserves a place on the reference shelves at the disposal of students. The usual list of useful quantities is appended, and there is a short appendix on conics.

**A Text-Book of Thermodynamics, with special reference to Chemistry.** By J. R. PARTINGTON. Pp. viii + 544. 14s. net. 1913. (Constable.)

Those who have had a taste of Mr. Partington's quality, as shown in his *Higher Mathematics for Chemical Students*, will turn with an added interest to the above volume, in which he deduces " the principles of Thermo-dynamics in the simplest possible manner, from the two fundamental laws," and illustrates " their applicability by means of a selection of examples." We are not here concerned with the fact that there are chemists of great repute who do not consider that reasoning based on thermodynamic principles can be invoked to advantage, and to whom the " manœuvres of the mathematically minded " are repulsive. The " intrusion " of mathematics into the realm of physical chemistry is, however, a *fait accompli*. The discussion as to whether the chemist should be taught higher mathematics may now be considered as closed. There may still be room for divergence of opinion as to how much the young chemist of the future requires, and as to the manner in which instruction should be given. And in this connection it will not be out of place to refer to the papers by Mr. Worley in *Science Progress* (January,

1912), on "Mathematics and Chemistry," and to the reply by Mr. Partington in the number for January of the following year. As for the volume before us, we may say at once that it amply fulfils our expectations. Here and there will be differences of opinion as to the propriety of the arrangement, but there will be none as to the author's lucidity of exposition, which is conspicuous in, for instance, such sections as those dealing with the phenomena of those doldrums—the regions of heterogeneous equilibria. A striking feature is the carefully selected references to the literature of the subject, both in the text and in bibliographical form, at the end of some of the chapters. The last two chapters will be welcomed by many teachers, containing as they do a succinct sketch of modern developments of the science, but they will leave many unsatisfied and eager for more—perhaps a not-altogether-to-be-regretted frame of mind. Thirty pages are devoted to the famous seven-year-old *Nernstsche Wärmetheorem* and its various applications. The final sections deal with Kinetic Theories in Thermodynamics, bringing us up to the *Theory of the Solid State*, published by Debye at the end of last year. It is interesting to note that the author describes the theory in question as "the most complete and successful attempt to represent the thermal state of solids which has yet been made by the aid of the theory of ergonic distribution. The book is attractively produced, and may be warmly recommended to teachers on the look out for a text-book for students of physical chemistry.

## THE LIBRARY.

The Librarian acknowledges with thanks the receipt of *A Study of Mathematical Education*, Benchara Branford, presented by the author; also from the Clarendon Press, *General Course of Pure Mathematics*, by A. L. Bowley, and *Elementary Statics*, by R. S. Heath.

The *Mathematical Teacher* is now being sent to the Library, and should have been acknowledged sooner—also a volume of the *Proceedings of the Royal Society of Upsala*.

The Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

1 or 2 copies of *Gazette* No. 2 (very important).
1 or 2 „ „ No. 8.
2 or 3 copies of Annual Report No. 11 (very important).
1 or 2 „ „ Nos. 10, 12 (very important).
1 copy „ Nos. 1, 2.

## FOR SALE.



GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.